Neural networks

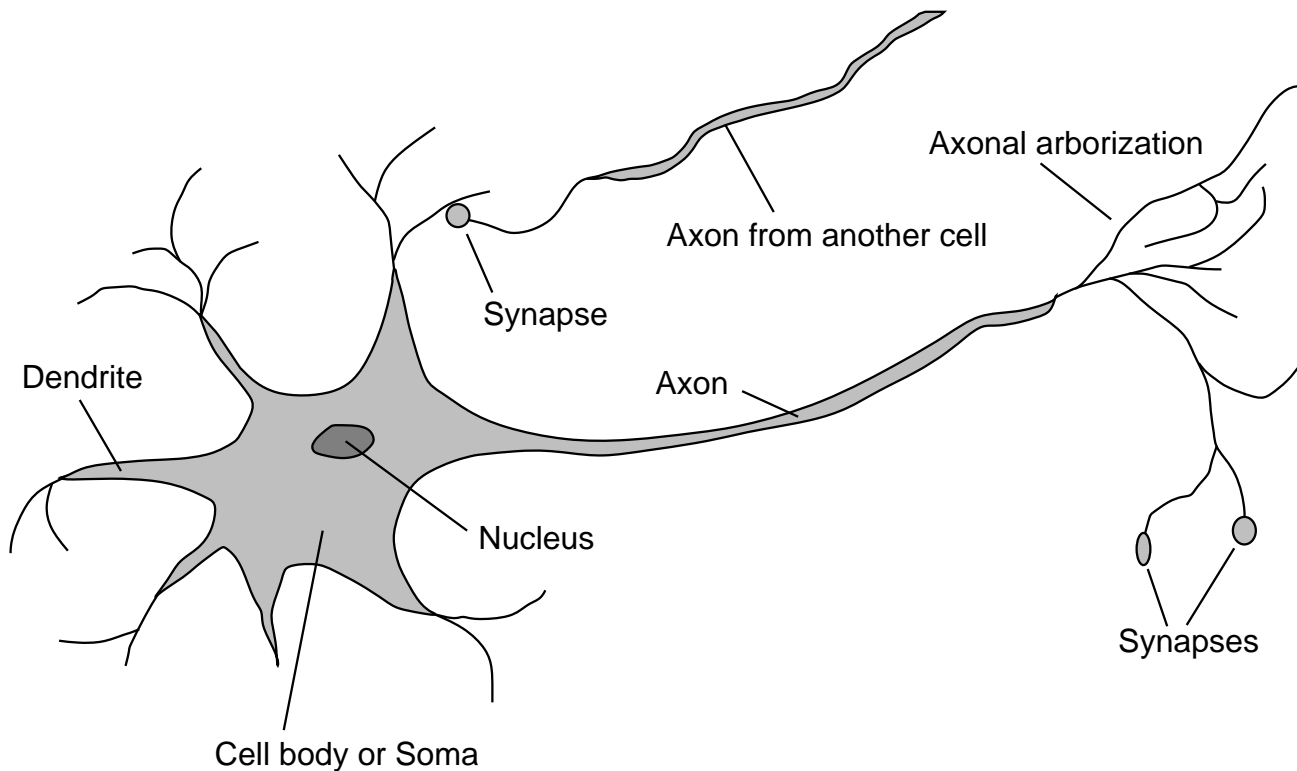## Chapter 20, Section 5

Outline

◇ Brains

◇ Neural networks

◇ Perceptrons

◇ Multilayer perceptrons

◇ Applications of neural networks

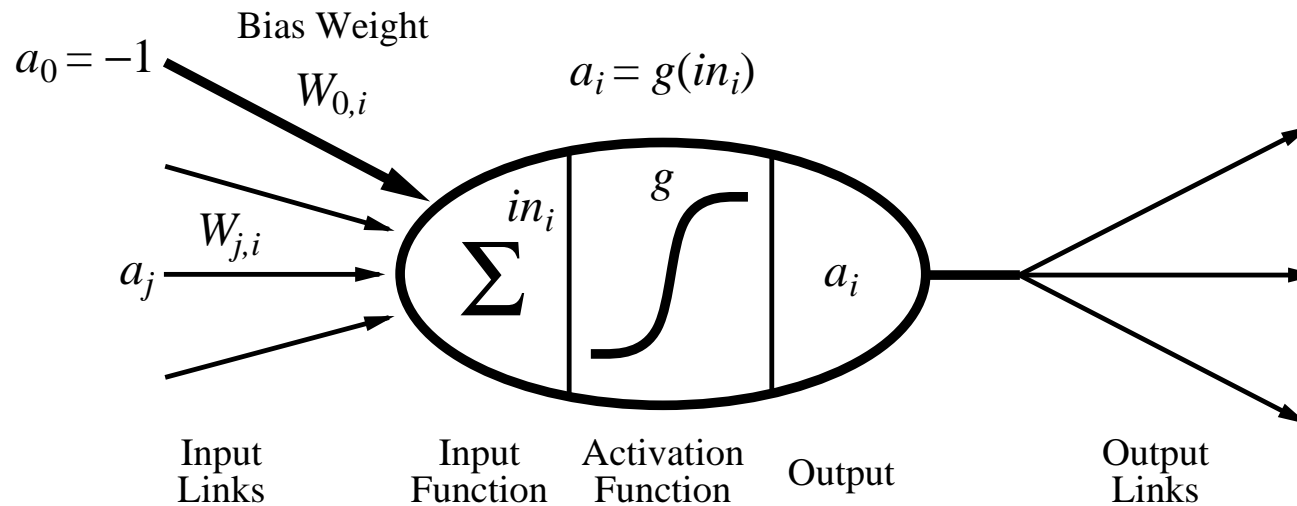# Brains

10^{11} neurons of > 20 types, 10^{14} synapses, 1ms–10ms cycle time
Signals are noisy "spike trains" of electrical potential

Axonal arborization

Axon from another cell

Synapse

Dendrite

Axon

Nucleus

Synapses

Cell body or Soma
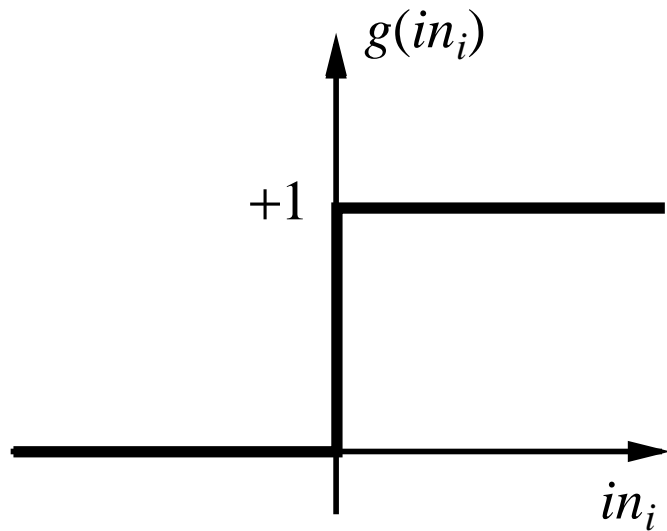
McCulloch–Pitts "unit"

Output is a "squashed" linear function of the inputs:

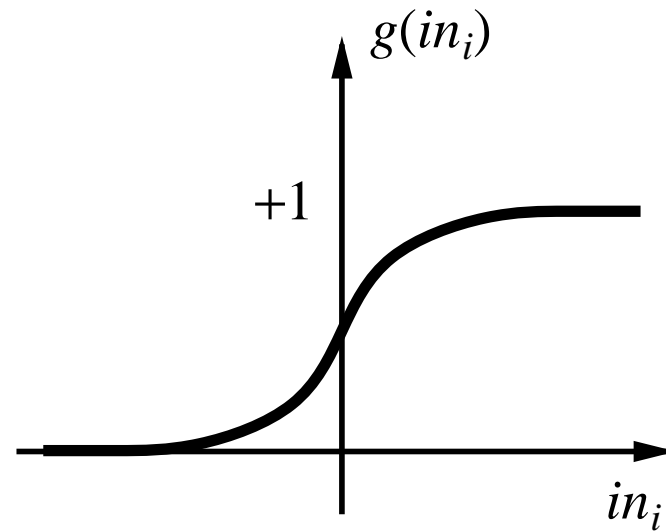$$a_i \leftarrow g(in_i) = g\left(\Sigma_j W_{j,i} a_j\right)$$

A gross oversimplification of real neurons, but its purpose is
to develop understanding of what networks of simple units can do
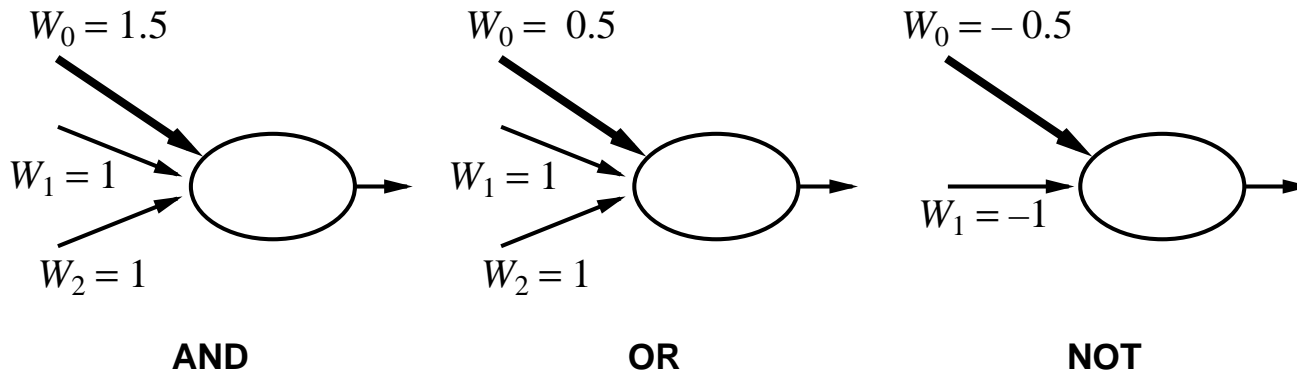
Activation functions

(a)

(b)

(a) is a step function or threshold function

(b) is a sigmoid function $1/(1 + e^{-x})$

Changing the bias weight $W_{0,i}$ moves the threshold location

Implementing logical functions

$W_0 = 1.5$

$W_1 = 1$

$W_2 = 1$

AND

$W_0 = 0.5$

$W_1 = 1$

$W_2 = 1$

OR

$W_0 = -0.5$

$W_1 = -1$

NOT

McCulloch and Pitts: every Boolean function can be implemented

Network structures

Feed-forward networks:
- – single-layer perceptrons
- – multi-layer perceptrons

Feed-forward networks implement functions, have no internal state

Recurrent networks:
- – Hopfield networks have symmetric weights $(W_{i,j} = W_{j,i})$
  $g(x) = \mathsf{sign}(x)$, $a_i = \pm 1$; **holographic associative memory**
- – Boltzmann machines use stochastic activation functions,
  $\approx$ MCMC in Bayes nets
- – recurrent neural nets have directed cycles with delays
  $\Rightarrow$ have internal state (like flip-flops), can oscillate etc.
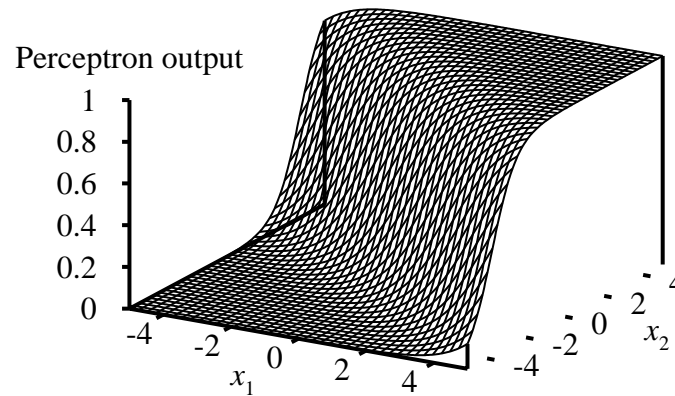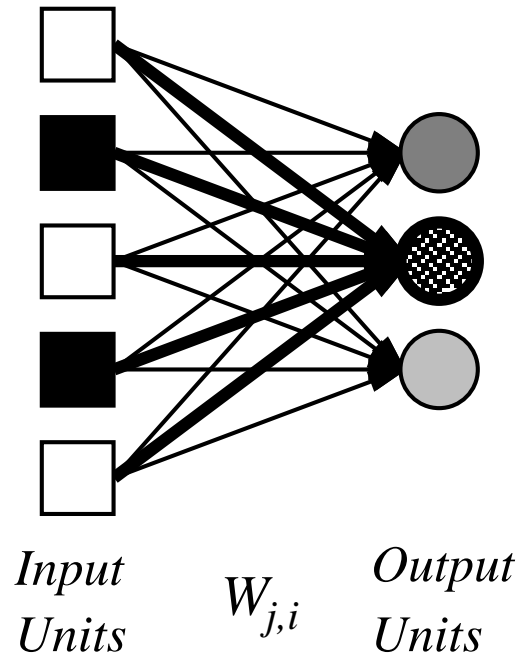
Feed-forward example
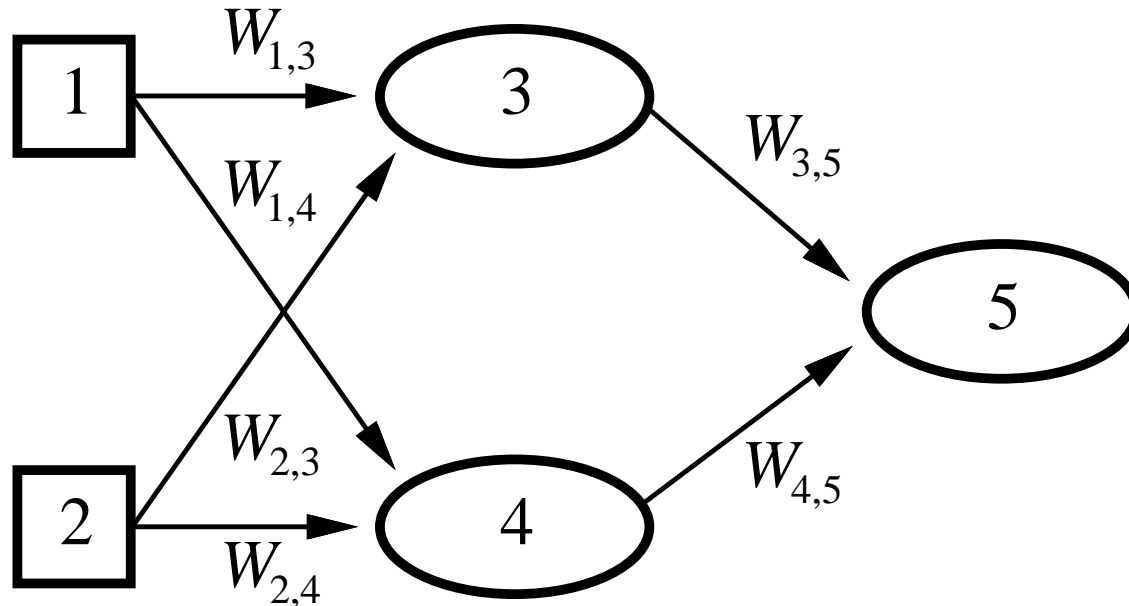
Feed-forward network = a parameterized family of nonlinear functions:

$$
\begin{aligned}
a_5 &= g(W_{3,5} \cdot a_3 + W_{4,5} \cdot a_4) \\
&= g(W_{3,5} \cdot g(W_{1,3} \cdot a_1 + W_{2,3} \cdot a_2) + W_{4,5} \cdot g(W_{1,4} \cdot a_1 + W_{2,4} \cdot a_2))
\end{aligned}
$$

Adjusting weights changes the function: do learning this way!

Single-layer perceptrons

Input Units

$W_{j,i}$

Output Units

Perceptron output

Output units all operate separately—no shared weights

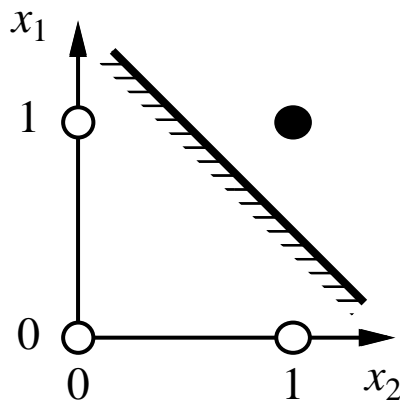Adjusting weights moves the location, orientation, and steepness of cliff

Expressiveness of perceptrons

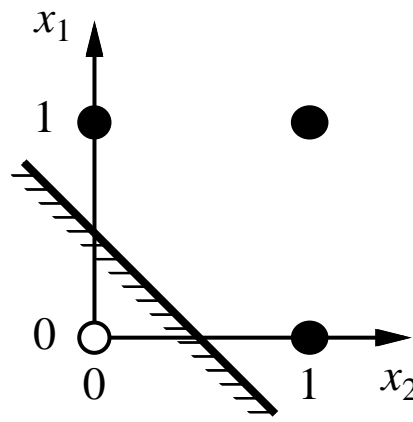Consider a perceptron with g = step function (Rosenblatt, 1957, 1960)

Can represent AND, OR, NOT, majority, etc., but not XOR

Represents a linear separator in input space:

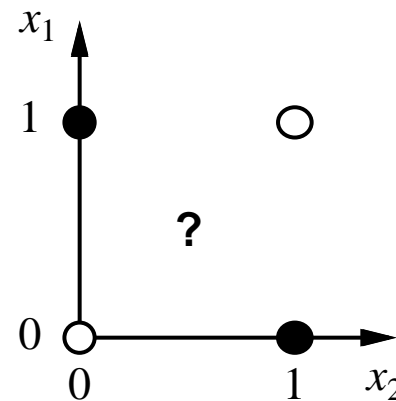$$\Sigma_j W_j x_j > 0 \quad \text{or} \quad \mathbf{W} \cdot \mathbf{x} > 0$$

(a) x_1 and x_2 (b) x_1 or x_2 (c) x_1 xor x_2

Minsky & Papert (1969) pricked the neural network balloon

Perceptron learning

Learn by adjusting weights to reduce error on training set

The squared error for an example with input $\mathbf{x}$ and true output y is

$$E = \frac{1}{2} Err^2 \equiv \frac{1}{2}(y - h_{\mathbf{W}}(\mathbf{x}))^2 ,$$

Perform optimization search by gradient descent:

$$\frac{\partial E}{\partial W_j} = Err \times \frac{\partial Err}{\partial W_j} = Err \times \frac{\partial}{\partial W_j}\left(y - g(\Sigma_{j=0}^{n} W_j x_j)\right)$$

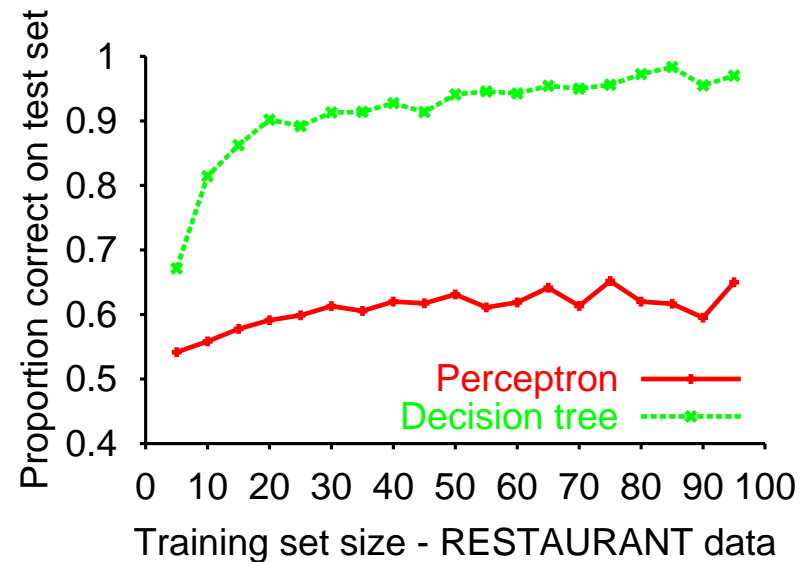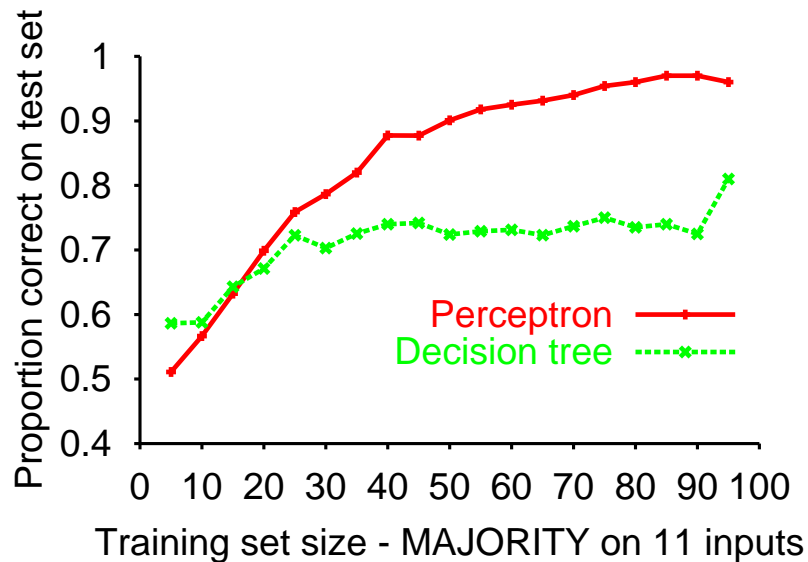$$= -Err \times g'(in) \times x_j$$

Simple weight update rule:

$$W_j \leftarrow W_j + \alpha \times Err \times g'(in) \times x_j$$

E.g., +ve error $\Rightarrow$ increase network output
$\Rightarrow$ increase weights on +ve inputs, decrease on -ve inputs

Perceptron learning contd.

Perceptron learning rule converges to a consistent function
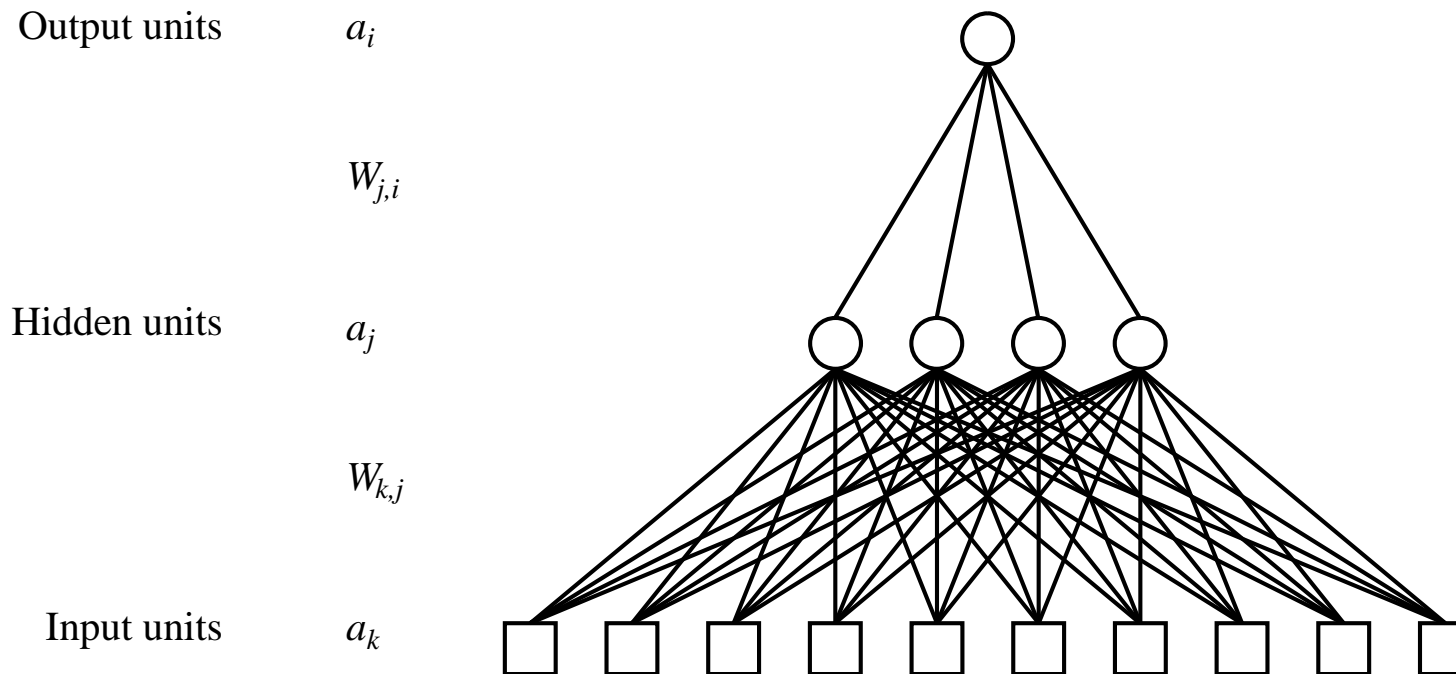**for any linearly separable data set**

Perceptron learns majority function easily, DTL is hopeless

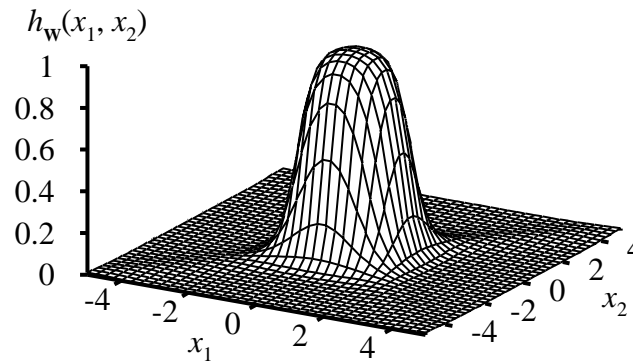DTL learns restaurant function easily, perceptron cannot represent it

Multilayer perceptrons

Layers are usually fully connected;
numbers of hidden units typically chosen by hand

Output units a_i

$W_{j,i}$

Hidden units a_j

$W_{k,j}$

Input units a_k

Expressiveness of MLPs

All continuous functions w/ 2 layers, all functions w/ 3 layers
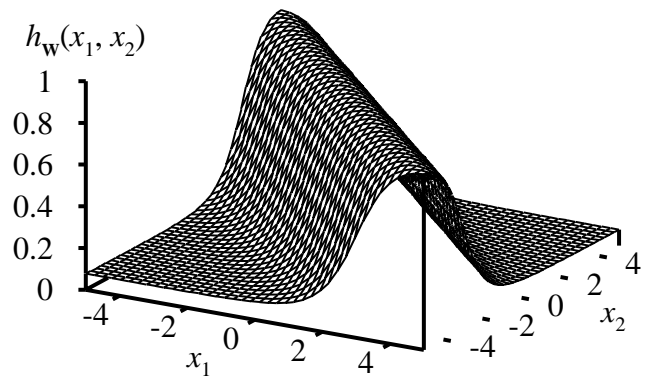
Combine two opposite-facing threshold functions to make a ridge

Combine two perpendicular ridges to make a bump

Add bumps of various sizes and locations to fit any surface

Proof requires exponentially many hidden units (cf DTL proof)

Back-propagation learning

Output layer: same as for single-layer perceptron,

$$W_{j,i} \leftarrow W_{j,i} + \alpha \times a_j \times \Delta_i$$

where $\Delta_i = Err_i \times g'(in_i)$

Hidden layer: **back-propagate** the error from the output layer:

$$\Delta_j = g'(in_j) \sum_i W_{j,i} \Delta_i \; .$$

Update rule for weights in hidden layer:

$$W_{k,j} \leftarrow W_{k,j} + \alpha \times a_k \times \Delta_j \; .$$

(Most neuroscientists deny that back-propagation occurs in the brain)

Back-propagation derivation

The squared error on a single example is defined as

$$E = \frac{1}{2} \sum_i (y_i - a_i)^2 \ ,$$

where the sum is over the nodes in the output layer.

$$
\begin{aligned}
\frac{\partial E}{\partial W_{j,i}} &= -(y_i - a_i)\frac{\partial a_i}{\partial W_{j,i}} = -(y_i - a_i)\frac{\partial g(in_i)}{\partial W_{j,i}} \\
&= -(y_i - a_i)g'(in_i)\frac{\partial in_i}{\partial W_{j,i}} = -(y_i - a_i)g'(in_i)\frac{\partial}{\partial W_{j,i}}\left(\sum_j W_{j,i}a_j\right) \\
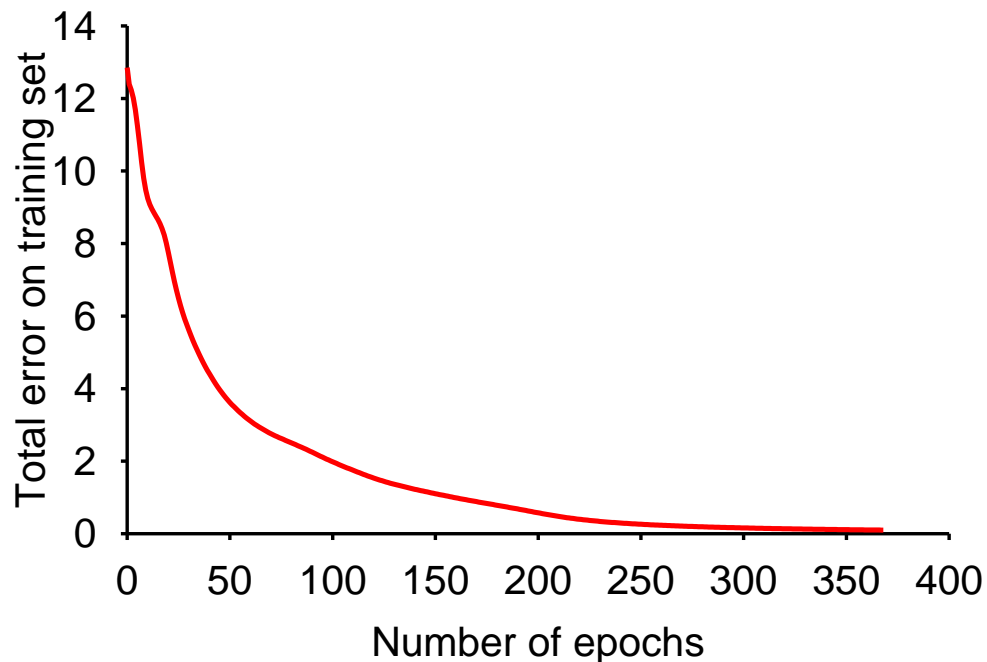&= -(y_i - a_i)g'(in_i)a_j = -a_j \Delta_i
\end{aligned}
$$

Back-propagation derivation contd.

$$\frac{\partial E}{\partial W_{k,j}} = -\sum_i (y_i - a_i)\frac{\partial a_i}{\partial W_{k,j}} = -\sum_i (y_i - a_i)\frac{\partial g(in_i)}{\partial W_{k,j}}$$

$$= -\sum_i (y_i - a_i)g'(in_i)\frac{\partial in_i}{\partial W_{k,j}} = -\sum_i \Delta_i \frac{\partial}{\partial W_{k,j}}\left(\sum_j W_{j,i}a_j\right)$$

$$= -\sum_i \Delta_i W_{j,i}\frac{\partial a_j}{\partial W_{k,j}} = -\sum_i \Delta_i W_{j,i}\frac{\partial g(in_j)}{\partial W_{k,j}}$$

$$= -\sum_i \Delta_i W_{j,i}g'(in_j)\frac{\partial in_j}{\partial W_{k,j}}$$

$$= -\sum_i \Delta_i W_{j,i}g'(in_j)\frac{\partial}{\partial W_{k,j}}\left(\sum_k W_{k,j}a_k\right)$$

$$= -\sum_i \Delta_i W_{j,i}g'(in_j)a_k = -a_k\Delta_j$$

Back-propagation learning contd.

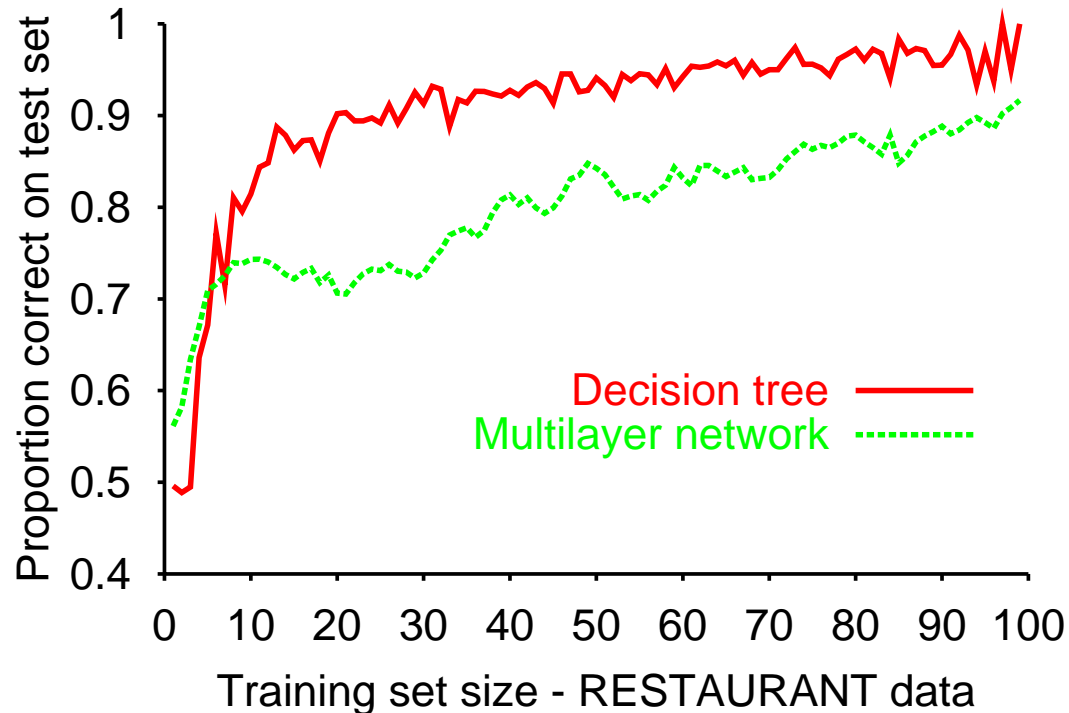At each epoch, sum gradient updates for all examples and apply

Training curve for 100 restaurant examples: finds exact fit

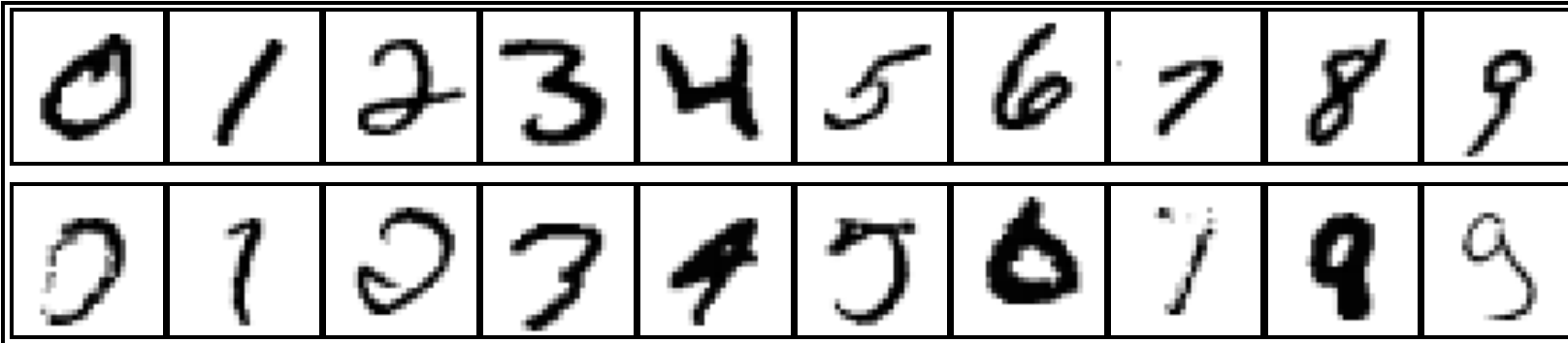Typical problems: slow convergence, local minima

Back-propagation learning contd.

Learning curve for MLP with 4 hidden units:

MLPs are quite good for complex pattern recognition tasks,
but resulting hypotheses cannot be understood easily

Handwritten digit recognition

3-nearest-neighbor = 2.4% error

400–300–10 unit MLP = 1.6% error

LeNet: 768–192–30–10 unit MLP = 0.9% error

Current best (kernel machines, vision algorithms) $\approx$ 0.6% error

Summary

Most brains have lots of neurons; each neuron $\approx$ linear–threshold unit (?)

Perceptrons (one-layer networks) insufficiently expressive

Multi-layer networks are sufficiently expressive; can be trained by gradient descent, i.e., error back-propagation

Many applications: speech, driving, handwriting, fraud detection, etc.

Engineering, cognitive modelling, and neural system modelling
subfields have largely diverged